Simple approach to the mesoscopic open electron resonator: Quantum current oscillations

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Abstract

The open electron resonator, described by Duncan et.al [1], is a mesoscopic device that has attracted considerable attention due to its remarkable behaviour (conductance oscillations), which has been explained by detailed theories based on the behaviour of electrons at the top of the Fermi sea. In this work, we study the resonator using the simple quantum quantum electrical circuit approach, developed recently by Li and Chen [2]. With this approach, and considering a very simple capacitor-like model of the system, we are able to theoretically reproduce the observed conductance oscillations. A very remarkable feature of the simple theory developed here is the fact that the predictions depend mostly on very general facts, namely, the discrete nature of electric charge and quantum mechanics; other detailed features of the systems described enter as parameters of the system, such as capacities and inductances.

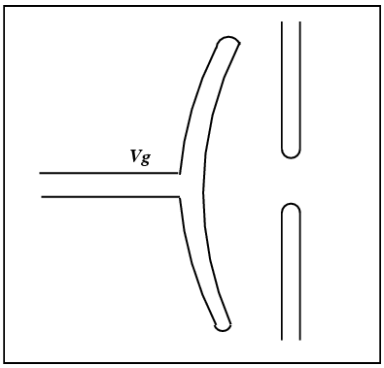

Figure 1: A drawing of the open resonator of Duncan et. al. [1], showing the reflector (left) and the two point contacts.

1 Introduction

The field of mesoscopic physics deals with the frontier between classical and quantum physics. However, this frontier is not sharply defined, and every case needs special attention. Phenomena like persistent currents [3], Coulomb blockade, magneto-resistance fluctuacions and current magnification [4, 5, 9, 10] are usually studied in this area. The open electron resonator is a device that has attracted some interest recently [6, 7, 8]. It consists of an open cavity in a two-dimensional electron gas, into which electrons can be injected via a quantum point contact, as described by Duncan et. al [1], and shown in Figure 1. Some very interesting properties have been found for this device, such as the conductance peaks observed upon changing the cavity length by $\lambda_f/2$ (half the 2D Fermi wavelength). Recently, Duncan et. al [1] have presented a series of conductance measurements, showing that the measured conductance through the point contact possesses a series of maxima, as a function of the gate voltage V_G. These measurements have been performed both at zero magnetic field (B), and as a function of it.

In this work, we present an alternative approach that allows us to explain the observed conductance peaks, as a function of the gate voltage V_G, at zero magnetic field, $B = 0$. This approach is related to the analogy between the mesoscopic device and a quantum electrical circuit with charge discreteness, and it will be discussed in the next section. In section 3 we study the properties of the energy spectrum of our model.

2 Our model

We start by observing that the open resonator of reference [1] is similar to a three-conductor system, in which one of them, the gate, is kept at voltage V_G, while current flows between the other conductors. We complete our model

device by assigning to it a self-inductance L. Classically, the circuit may be described as a sort charged $L-C$ circuit. We assume then that a current I flows between the two charged conductors (with charges Q and $-Q$); further, for simplicity, assume a plane geometry, and compute the classical electrostatic energy, obtaining

$$U = C_1 V_g^2 + \frac{Q^2}{2C_2} + \alpha Q V_g, \tag{1}$$

where C_1 and C_2 are capacities of the system, and α is a geometry-related number of order unity. Note that this fact will enable us to obtain some rough numeric estimates, based on the physical dimensions of the device, which are in reasonable agreement with observations, as it will be shown later.

As it has been proposed by Li and Chen [2], and later by Flores [9], and Flores and Utreras [10], we treat our mesoscopic system as a quantum electrical circuit, with quantized charge. The classical Hamiltonian H_{cl} of the (circuit) model system, written in terms of the canonically conjugate variables Q and Φ is

$$H_{cl} = \frac{\Phi^2}{2L} + U = \frac{\Phi^2}{2L} + C_1 V_g^2 + \frac{Q^2}{2C_2} + \alpha Q V_g. \tag{2}$$

To quantize the system, the variables Q and Φ are replaced by the operators $\widehat{Q}$ and $\widehat{\Phi}$. Further, to recover the quantization of charge within this electrical circuit approach, we introduce the replacement [2, 9, 10]

$$\widehat{\Phi} \to \frac{2\hbar}{q_e} \sin\left(\frac{q_e}{2\hbar}\widehat{\Phi}\right), \tag{3}$$

where q_e is the quantum of charge. We remark that, if charge discreteness is neglected, the operator $\widehat{\Phi}$ may be directly identified with the magnetic flux operator, and therefore directly related to the current; however, when one introduces charge discreteness via the replacement above, the simple relation to the current is lost, therefore, after replacement (3) the flux operator becomes the pseudo-flux. This pseudo-flux operator satisfies the usual commutation relation$[\widehat{Q}, \widehat{\Phi}] = i\hbar$. Notice that, given the complexity of dealing with an operator such as the one above (3), it is simpler to work in the so-called pseudo-flux representation, in which the operator $\widehat{\Phi}$ is replaced by its eigenvalue ϕ, while the charge operator is given by $\widehat{Q} = i\hbar\partial/\partial\phi$. In this way, the resulting hamiltonian is given by

$$\widehat{H} = \frac{2\hbar^2}{q_e^2 L} \sin^2(q_e\widehat{\phi}/2\hbar) + C_1 V_g^2 + \frac{\widehat{Q}^2}{2C_2} + \alpha\widehat{Q}V_g. \tag{4}$$

The hamiltonian operator above constitutes our starting point, and our working hypothesis. The parameters of our theory, particularly L and C, are related to the geometry of the system, but are hard to compute for a given experimental system, therefore, they should be deduced from experimental observations.

3 Energy spectrum

To study the properties of our system, define first $\beta = (C_1 - \alpha^2 C/2)$, $C = C_2$, and $a = \alpha C V_g/\hbar$, and rewrite our hamiltonian as

$$\widehat{H} = \beta V_g^2 + \frac{2\hbar^2}{q_e^2 L}\sin^2(q_e\phi/2\hbar) + \frac{\hbar^2}{2C}(i\frac{d}{d\phi} + a)^2. \tag{5}$$

It is observed that, in this representation, the magnetic energy term acts like a periodic potential in the variable ϕ, while the electrostatic energy term here becomes a kind of 'kinetic' term, i.e., formally, the terms reverse their meaning, when one compares with a 'regular' Schrödinger equation. The term βV_g^2 is just a constant, and it will be eliminated from the hamiltonian. To make our analogies more clear, define a periodic 'potential energy function' $V(\phi)$, with period $\phi_0 = 2\pi\hbar/q_e = h/q_e$

$$V(\phi) = \frac{2\hbar^2}{q_e^2 L}\sin^2(q_e\phi/2\hbar) = \frac{\hbar^2}{q_e^2 L}\left\{1 - \cos(q_e\phi/\hbar)\right\}. \tag{6}$$

The Schrödinger equation $H\Psi = E\Psi$ may now be written as

$$\left\{V(\phi) + \frac{\hbar^2}{2C}(i\frac{d}{d\phi} + a)^2\right\}\Psi(\phi) = E\Psi(\phi). \tag{7}$$

The wavefunction $\Psi(\phi)$ satisfies the strong periodicity condition

$$\Psi(\phi + \phi_0) = \Psi(\phi), \tag{8}$$

this condition results from the fact that quantization in the charge space implies periodicity in the (pseudo) flux space, as shown by Li and Chen [2].

Observe that the hamiltonian $\widehat{H}$ above commutes with the discrete translation operator $\widehat{T}_{m\phi_0}\Psi(\phi) = \Psi(\phi + m\phi_0)$, for integer m; therefore, for any value of a, Bloch's theorem holds, specifically, the energy $E(a)$ should be a periodic function of a, labeled by a 'band index' n, further, the parameter a should play the role of the wavenumber in Bloch theory; the period for a is $\Delta a = q_e/\hbar$, and therefore, the energies are periodic functions of the gate voltage V_G, with period $\Delta V_g = q_e/\alpha C$ (recall that α is a number that we take as unity from here on).

To proceed with the numerical calculations, we should get some estimates of the parameters of the theory, namely, the values of the capacity C and the inductance L. These may be obtained from the knowledge of the linear dimensions of the system, i.e., of the order of $1\mu m$, and using the classical formula $C = \epsilon_0 lb/d$, in which l is taken as the lateral dimension of the cavity, d the distance of the contact points and b is the thickness of the 2D electron gas in which the device is immersed, now since $l \approx d$, and b is a fraction of $a\mu m$, we obtain $\Delta V_G \approx 0.02(Volts)$, a reasonable estimate of the effect. As to the estimate for L, it is a parameter related to the size of the circuit that contains the device, i.e., a relatively large number $L \approx \mu_0 D$. As a measure of this, we have taken the ratio of the magnetic potential energy, ϕ_0^2/L to the electric energy $q_e^2/2C$ to be of order unity, for the numerical calculations described here.

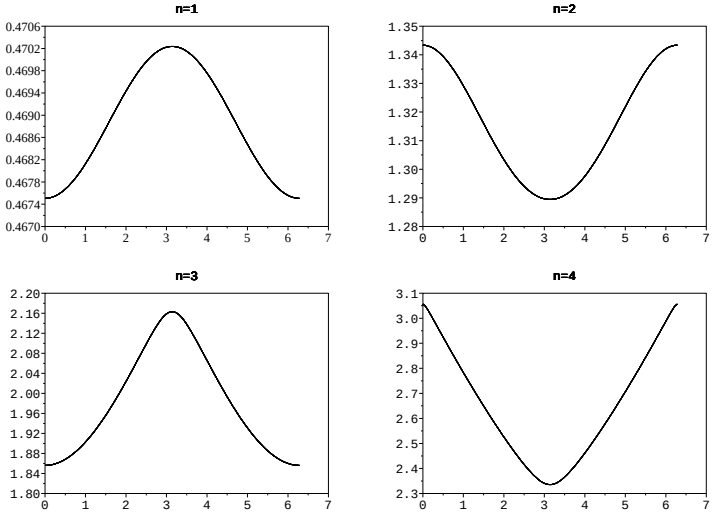

Figure 2: Computed energy eigenvalues (first bands) as a function of the gate voltage V_G. The ordinate is the energy, on the dimensionless energy scale $(\phi_0^2/2L)/(q_e^2/2C)$, the abcisa is the dimensionless parameter $a\phi_0 = (2\pi C/q_e)V_G$. Observe that the ground state 'band' has a very small oscillation amplitude, if shown on the same scale than the other curves, it would appear completely flat.

Notice that we have estimated the capacity of the system using the classical formula, but to include small-size corrections it may be more appropriate to obtain tha capacity from a microscopic theory, or, at least from Thomas-Fermi theory.

In Appendix A, we discuss the procedure used to find the energy eigenfunctions $u_E(\phi)$ and their corresponding energies for our hamiltonian. The solutions are expressed in terms of two normalized and mutually orthogonal functions $C_E(\phi)$ y $S_E(\phi)$. The eigenvalue conditions is expressed by eq.(21), which shows explicitly the periodicity of the energies as a function of V_G.

We have computed the eigenfunctions $u = u_E(\phi) = u_{n,V_G}(\phi)$, by numerically solving the eigenvalue equation (Eq. 21, in Appendix), for the lowest eigenstates, as a function of the parameter $a = CV_g/\hbar$. The full wavefunction $u(\phi)$ is given by the linear combination

$$u_E(\phi) = AC_E(\phi) + BS_E(\phi), \tag{9}$$

with coefficients given by

$$A = \frac{\Delta_S}{\sqrt{|\Delta_C|^2 + |\Delta_S|^2}}$$

$$B = -\frac{\Delta_C}{\sqrt{|\Delta_C|^2 + |\Delta_S|^2}}.$$

The results of our energy calculations are displayed in Figure 2, which shows that the energies are distributed in energy bands with the periodic structure already described (period $\Delta a = 2\pi/\phi_0$). Observe that the first band has very small width, which depends on the relation of electric to magnetic energy.

4 Conductance Oscillations

From a general point of view, we may expect the same periodicity in the conductance as that of the energy. If we assume A as the probability per unit of time of decaying transitions between two levels (n, n') then the emission power is $A(E^0(n) - E^0(n'))$, which must be proportional to the conductance G (the proportionality constant is the square of the external bias voltage). Since the energy spectrum is periodic in $\Delta V_g = q_e/\alpha C$, so is the conductance.

We have shown that the general features of the energy sprectrum do not depend on the value of the inductance L, in the sense that, as long as $L \neq 0$, the energy spectrum will display the V_G periodicity. However, for small values of L, this behaviour will not be seen, since the amplitude of the energy oscillations will be very small. This is actually observed (see figure 2) for the lowest energy 'band', even for moderate values of L, in other words, the details of the energy spectrum do depend on L, but its general properties, such as the periodicity depend only on the capacity of the system, as shown by the experiments.

We remark that our approach is able to explain the experimentally observed [1] oscillatory behaviour of G, as a function of V_G, but the linear dependence of the current on V_G, is not directly explained by our approach, or, to put it differently, the present theory does not describe the linear dependence of the current on the gate voltage. We observe that, for fixed V_G, we may change the capacity of the system, $C' = \alpha C_2$, by changing one of the linear dimensions of the cavity(l, say), while keeping constant both the distance from the contact points constant (d), and the other linear dimension (b); it is clearly seen then that the current is also periodic in l, i.e., since we may write for C'

$$C' = \epsilon_0 \alpha l b/d, \tag{10}$$

in this way, the oscillatory behaviour of the current (and, hence, of the conductance too) as a function of the resonator cavity length may also be explained.

Now we make an estimate of the size of the effect, as suggested by our theory. Note that in C' above, the area $l \times b$ is the product of the linear dimensions of the reflector, while d is the distance to the contact points. We see that two of these distances are of order $1\mu m$, therefore, $C' \approx \epsilon_0 \times b$ (b being the perpendicular dimension to the 2D electron gas, of order $1\mu m$, then $\Delta V_g \approx 0.02(V)$, a pretty good answer, given the rough estimate for the capacity C' used here.

5 Discussion

This article offers an alternative point of view on the open electron resonator, while recognizing that there exists an accepted explanation of the effect. Nevertheless, our alternative description is very simple, powerful and it does not require elaborate assumptions. It is a new description in terms of elements like capacitors and inductances, which is not too widely different from more usual treatments for mesoscopic systems [13].

It is also worth considering the issue of the spatial dimensionality of the theory. A casual glance of our equations gives one the impression that the theory is one-dimensional. This is not the case, since the theory is expressed in terms of charge-flux variables, and no reference is made to the dimensionality of the devices it describes; in fact, such spatial dimensionality only appears through the parameters of the theory, such as capacities and inductances, just as it happens in classical electrical circuits. It is worth noticing that such theory is more correctly described as "topological", and not "geometrical. Therefore, a correct use of the theory may help in sheding some light in other mesoscopic systems in study today.

6 Acknowledgements

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A Solution of the Eigenvalue equation

Consider now the (numerical) solution of the resonator equation (7), for the eigen-function $\Psi(\phi)$ and eigen-energies, as a function of the parameter V_G (a), with the periodic boundary conditions (8). As indicated previously, the wavefunction $\Psi(\phi)$ may be written as

$$\Psi(\phi) = e^{ia\phi}u(\phi)$$

This replacement has the property that it eliminates the constant $'a'$ from the Schrödinger equation for the unknown function $u(\phi)$. This wavefunction is not periodic, but it satisfies a simple Bloch-type boundary condition (which follows from the periodicity of Ψ).

$$u(\phi + \phi_0) = e^{-ia\phi_0}u(\phi) \tag{11}$$

Now, to actually find numerical solutions of the Schrödinger equation, and their corresponding energies, we proceed as follows. For each value of the energy E, we find two independent numerical solutions, which we call $C(\phi)$ and $S(\phi)$.

For example, for a fundamental period with endpoints at $\phi = 0$ and $\phi = \phi_0$, these functions may be chosen to satisfy

$$C_E(\phi_0/2) = 1 \quad C'_E(\phi_0/2) = 0 \tag{12}$$
$$S_E(\phi_0/2) = 0 \quad S'_E(\phi_0/2) = 1 \tag{13}$$

The wavefunction u may now be expressed as the linear combination

$$u_E(\phi) = A \cdot C_E(\phi) + B \cdot S_E(\phi). \tag{14}$$

The boundary conditions are now

$$\begin{aligned} u_E(\phi_0) &= e^{-ia\phi_0} u_E(0) \\ u'_E(\phi_0) &= e^{-ia\phi_0} u'_E(0), \end{aligned} \tag{15}$$

which lead to two linear equations for the coefficients A and B,

$$\begin{aligned} \Delta_C A + \Delta_S B &= 0 \\ \Delta'_C A + \Delta'_S B &= 0, \end{aligned} \tag{16}$$

in which the coefficients are

$$\begin{aligned} \Delta_C &= e^{ia\phi_0/2} C(\phi_0) - e^{-ia\phi_0/2} C(0) \\ \Delta_S &= e^{ia\phi_0/2} S(\phi_0) - e^{-ia\phi_0/2} S(0) \\ \Delta'_C &= e^{ia\phi_0/2} C'(\phi_0) - e^{-ia\phi_0/2} C'(0) \\ \Delta'_S &= e^{ia\phi_0/2} S'(\phi_0) - e^{-ia\phi_0/2} S'(0) \end{aligned} \tag{17}$$

The eigenvalue equation for the energies is given by the vanishing of the determinant of the coefficients,

$$G(E) = \Delta_C \Delta'_S - \Delta_S \Delta'_C = 0 \tag{18}$$

When, as it happens in our case, the potential is symmetric with respect to the midpoint of the interval, the functions C and S have definite parity about that point, i.e. $C(\phi_0) = C(0)$ and $S(\phi_0) = -S(0)$, the derivatives have the opposing parity, $C'(\phi_0) = -C'(0)$ and $S'(\phi_0) = S'(0)$, therefore, defining $\theta = a\phi_0/2$, we obtain a simplified eigenvalue equation,

$$G(E) = \sin^2(a\phi_0/2) C(0) S'(0) + \cos^2(a\phi_0/2) S(0) C'(0) = 0 \tag{19}$$

This equation is convenient for numerical work, since the functions C and S are easily computed numerically; to find the eigenvalues E, we fix a and compute $G(E)$, the zeros correspond to the eigenstates.

For the general case (non symmetric), it is useful to define the coefficients

$$
\begin{aligned}
C_{(-)} &= C(\phi_0) - C(0) \\
C_{(+)} &= C(\phi_0) + C(0) \\
S_{(-)} &= S(\phi_0) - S(0) \\
S_{(+)} &= S(\phi_0) + S(0) \\
C'_{(-)} &= C'(\phi_0) - C'(0) \\
C'_{(+)} &= C'(\phi_0) + C'(0) \\
S'_{(-)} &= S'(\phi_0) - S'(0) \\
S'_{(+)} &= S'(\phi_0) + S'(0).
\end{aligned}
$$

With this, the coefficients of the eigenvalue equation become

$$
\begin{aligned}
\Delta_C &= C_{(-)}\cos(\theta) + iC_{(+)}\sin(\theta) \\
\Delta_S &= S_{(-)}\cos(\theta) + iS_{(+)}\sin(\theta) \\
\Delta'_C &= C'_{(-)}\cos(\theta) + iC'_{(+)}\sin(\theta) \\
\Delta'_S &= S'_{(-)}\cos(\theta) + iS'_{(+)}\cos(\theta)
\end{aligned} \tag{20}
$$

The eigenvalue equation is purely real (its imaginary part $\Im[G(E)] = 0$ vanishes identically since the Wronskian -$W[C,S]$- of two solutions is a constant):

$$
G(E) = \Big(C_{(-)}S'_{(-)} - S_{(-)}C'_{(-)}\Big)\cos^2(\theta) + \Big(S_{(+)}C'_{(+)} - C_{(+)}S'_{(+)}\Big)\sin^2(\theta). \tag{21}
$$

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